Large Compound Lotteries*

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Abstract

Extending preferences over simple lotteries to compound (two-stage) lotteries can be done using two different methods: (1) using the Reduction of compound lotteries axiom, under which probabilities of the two stages are multiplied; (2) using the compound independence axiom, under which each second stage lottery is replaced by its certainty

compound lotteries axiom (RCLA), that is, by multiplying the probabilities of the various final outcomes. For example, if X = (; x_i p_i;) and Y = (; $_j$ q_i;) are lotteries, then the compound lottery = (X ; Y 1$-$) is viewed as X + (1$-$)Y = (; x_i p_i; $_j$ (1$-$)q_j;) (see Samuelson [27]). De Finetti's [13] went even further, claiming that probabilities over probabilities are just probabilities, therefore such compound lotteries do not take us out of the original space of lotteries. Denote by $_{R}$ the reduced form of the compound lottery using RCLA.

Alternatively, one can use the compound independence axiom (CIA) to reduce compound lotteries recursively, where = (X ; Y 1$-$) is assumed to be indi erent to the simple lottery ((X) ; (Y) 1$-$) over the certainty equivalents of X and Y. Kreps and Porteus [18] and Segal [28] presented a formal analysis of this procedure. Numerous experiments show this is the way many decision makers view compound lotteries and that RCLA is widely violated. See, e.g., Halevy [16], Chew, Miao, and Zhong [10], Gillen, Snowberg, and Yariv [14], Abdellaoui, Klibano , and Placido [1], and Epstein and Halevy [11]. Denote by $_{CI}$ the reduced form of using CIA.

If preferences are expected utility, then for all , $_{R} \sim$ $_{CI}$ and expected utility is the only theory to have this property. Moreover, each of the two methods without the other may seem to violate some kind of monotonicity. For example, suppose that X is indi erent to Y, yet $\frac{1}{2}X + \frac{1}{2}Y$ is preferred to both, hence ($\frac{1}{2}X + \frac{1}{2}Y$) ($X$) = ($Y$). Consider the compound lotteries = (X $\frac{1}{2}$; Y $\frac{1}{2}$) and = ($\frac{1}{2}$($X -$) + $\frac{1}{2}Y$ 1) where 0 is su ciently small so that ($\frac{1}{2}$($X -$) + $\frac{1}{2}Y$) (X) = (Y). Then (($\frac{1}{2}$($X -$) + $\frac{1}{2}Y$) 1) dominates ((X) $\frac{1}{2}$; (Y) $\frac{1}{2}$) by first-order stochastic dominance, yet $\frac{1}{2}X$ + $\frac{1}{}$

two rounds will reduce it to $p_2 \nearrow \quad p_1$

Segal [28]).

Reduction of Compound Lotteries Axiom (RCLA) For all $\in \mathcal{Q}$,

$$\sim \ _{\text{R}} := (\quad ; x_{i,j} \ q_i p_{i,j}; \quad)$$

Compound Independence Axiom (CIA) For all $\in \mathcal{Q}$,

$$\sim \ _{\text{CI}} := (\ (X_1) \ q_1; \quad ; \ (X_m) \ q_m)$$

where (X), the certainty equivalent of X, is given by $_{c(X)} = (\ (X) \ 1) \sim X$.

Consider now the case in which n replicas of $= (X_1 \ q_1; \quad ; X_m \ q_m) \in \mathcal{Q}$ are simultaneously played. Let ^{n} be the two stage lottery where the first stage determines for each which lottery X_i will be played in the second stage. This is done for each lottery independently of the other lotteries. In the second stage, the decision maker is facing the sum of n lotteries, each taken from the set $\{X_1 \quad X_m\}$. There are $H := {}^n$ (to the power of n) such possible sequences, denote their sums Y_{nj}, $= 1 \quad H$, with the corresponding probabilities $_{nj}$, which are the product of the corresponding q_i probabilities. Observe that being the sum of simple lotteries, each Y_{nj} is a simple lottery. We thus obtain that

$$^n = (Y_{n1} \quad _{n1}; \quad ; Y_{nH} \quad _{nH}) \tag{1}$$

The two-stage lottery ^{n} yields the lotteries Y_{nj} with probabilities $_{nj}$, $= 1 \quad nH$. For example, let $X_1 = (-1 \ \frac{1}{2}; 0 \ \frac{1}{2})$, $X_2 = (-3 \ \frac{1}{2}; 0 \ \frac{1}{2})$, $= (X_1 \ \frac{1}{2}; X_2 \ \frac{1}{2})$, and $n = 2$. The four possible sequences are $Y_{21} = X_1 + X_1 = (-2 \ \frac{1}{4}; -1 \ \frac{1}{2}; 0 \ \frac{1}{4})$, $Y_{22} = X_{23} = X_1 + X_2 = X_2 + X_1 = (-4 \ \frac{1}{4}; -3 \ \frac{1}{4}; -1 \ \frac{1}{4}; 0 \ \frac{1}{4})$, $Y_{24} = X_2 + X_2 = (-6 \ \frac{1}{4}; -3 \ \frac{1}{2}; 0 \ \frac{1}{4})$, and $^2 = (Y_{21} \ \frac{1}{4}; \quad ; Y_{24} \ \frac{1}{4})$.

The lottery $(\ ^n)_{\text{R}}$ is obtained by taking the weighted mixture of these lotteries, that is, $_j \ _{nj} Y_{nj}$. The lottery $(\ ^n)_{\text{CI}}$ is obtained by replacing each Y_{nj} with its certainty equivalent. For simplicity, we denote them $^n_{\text{R}}$ and $^n_{\text{CI}}$.

3 Main Result

Our analysis depends on a technical wrapping assumption which we later show to be satisfied by most theories in the literature under conditions that can easily be justified.

Wrapping: **A preference relation $\succeq$ satisfies wrapping if it can be represented by a functional with the following property: There exist $\geqslant 1$ and**

$\geqslant$

Theorem 1 assumes that the expected utility (with respect to $\ _v$) of $\bar{\ }_{\mathbf{R}}$ is different from that of $\hat{\ }_{\mathbf{R}}$. This does not mean that if E[$_v(\bar{\ }_{\mathbf{R}})$] = E[$_v(\hat{\ }_{\mathbf{R}})$], then there exists n^* such that for all $n \geqslant n^*$, $\bar{\ }^n_{\mathbf{R}} \sim \hat{\ }^n_{\mathbf{R}}$ and $\bar{\ }^n_{\mathbf{CI}} \sim \hat{\ }^n_{\mathbf{CI}}$, or even that $\bar{\ }^n_{\mathbf{R}} \succeq \hat{\ }^n_{\mathbf{R}}$ iff $\bar{\ }^n_{\mathbf{CI}} \succeq \hat{\ }^n_{\mathbf{CI}}$. The reason is that unless one sequence is increasing and the other decreasing, the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ doesn't imply any specific relation between a_n and b_n (see the proof of the theorem).

On the other hand, consider a compound lottery $\ \in \tilde{\mathcal{Q}}$. The set of lotteries $\ _{\mathbf{R}}$ such that E[$_v(\ _{\mathbf{R}})$] $\neq$ E[$_v(\ _{\mathbf{R}})$] is open and dense in $\mathcal{X}$. In other words, if E[$_v(\ _{\mathbf{R}})$] $\neq$ E[$_v(\ _{\mathbf{R}})$], then this inequality holds for all sufficiently small perturbations of and , and if E[$_v(\ _{\mathbf{R}})$] = E[$_v(\ _{\mathbf{R}})$], then almost all small perturbations of either or will break this equality.

Given two lotteries $\bar{\ }$ and $\hat{\ }$, Theorem 1 needs to know the shape of v as $x \to -\infty$. But if $\bar{\ }_{\mathbf{R}}$ dominates $\hat{\ }_{\mathbf{R}}$ by first-order stochastic dominance, then regardless of the exact form of v, the expected utility of $\bar{\ }_{\mathbf{R}}$ is higher than that of $\hat{\ }_{\mathbf{R}}$. Therefore, not only is $\bar{\ }^n_{\mathbf{R}}$ preferred to $\hat{\ }^n_{\mathbf{R}}$ for all n, but for a sufficiently large n, $\bar{\ }^n_{\mathbf{CI}}$ is also preferred to $\hat{\ }^n_{\mathbf{CI}}$. Formally:

Conclusion 1 If $\bar{\ }_{\mathbf{R}}$ first-order stochastically dominates $\hat{\ }_{\mathbf{R}}$, then for every $\succeq$ satisfying the assumptions of Theorem 1 and for sufficiently large n, $\bar{\ }^n_{\mathbf{R}} \succ \hat{\ }^n_{\mathbf{R}}$ and $\bar{\ }^n_{\mathbf{CI}} \succ \hat{\ }^n_{\mathbf{CI}}$.

than differentiability and implies weak Gâteaux differentiability (see [9]).[5]

Claim 1 If the preference relation $\succeq$ can be represented by an RDU functional where is Lipschitz, then it satisfies wrapping.

Violations of the assumption that is Lipschitz lead to doubtful behavior. Since is concave, being non-Lipschitz implies that $\lim_{\varepsilon \to 0} \frac{g(\varepsilon)}{\varepsilon} = \infty$. For a given outcome

Weighted Utility

The WU model (see Chew [7]) is given by

$$(Y) = \frac{w(t)}{w(t)\mathrm{d}F_Y(t)} \cdot v(t)\mathrm{d}F_Y(t)$$

where w is continuous and zero is not in its image, hence wlg, w 0. We assume that ($_z$) = v( ) belongs to $\mathcal{U}$. Chew [7, eq. (5.2)] showed that $-\frac{w'}{w}$ increases the meas

Quadratic Utility

The general quadratic model is of the form

the inequality $(Y_k) \geqslant \mathrm{E}[v(Y_k)] - v(\bar{\ })$ fails to hold for sufficiently large. Theorem 2 below shows that even though the functional of eq. (4) does not satisfy wrapping, the conclusion of Theorem 1 holds for the CU model as well.

Let $\succeq$ be CU where $\mathcal{W} \in \Omega$ is generated by the utility functions $u_1 \ldots u_\ell$. Let $\bar{a} := \max_j \{a_{u_j}\}$ where $a_{u_j} = \lim_{x \to -\infty} -\frac{u_j''(x)}{u_j'(x)}$, and let $_{\mathcal{W}}(x) = -e^{-\bar{a}x}$.

Theorem 2 Suppose that the preference relation $\succeq$ is CU where $\mathcal{W} \in \Omega$ and let $\hat{\ } \in \tilde{\mathcal{Q}}$. If $\mathrm{E}[_{\mathcal{W}}(\bar{\ }_{\mathrm{R}})]\ \mathrm{E}[_{\mathcal{W}}(\hat{\ }_{\mathrm{R}})]$, then there exists n^* such that for all $n \geqslant n^*$, $^n_{\mathrm{R}} \succ \hat{\ }^n_{\mathrm{R}}$ and $^n_{\mathrm{CI}} \succ \hat{\ }^n_{\mathrm{CI}}$.

Theorem 2 assumes that $\mathcal{W}$ is generated by a finite set of utility functions. But it can be extended to the case where all the generating functions exhibit (weakly) decreasing absolute risk aversion, even if this set of functions is not finite, provided $\bar{a} = \sup\{a_u : u \in \mathcal{W}\} < \infty$.

Another case is Gul's [15] model of disappointment aversion. Cerreia-Vioglio, Dilleberger, and Ortoleva [5] show that it is a CU model where $\mathcal{W}$ is the family of its local utilities. This set is not the convex hull of a finite number or utilities (unless $b = 0$), and as these local utilities are not differentiable, they are not in $\mathcal{U}$, yet this model satisfies the conclusion of Theorem 2 (see Claim 3 above).

6 Dutch Books

Violating any of the two methods for analyzing two-stage lotteries exposes decision makers to Dutch books. De Finetti [12] claims that a decision maker whose preferences violate the basic laws of probability theory is exposed to manipulations that inevitably will lose him money. Markowitz [22] and Raiffa [25] presented arguments against changing preferences while moving down a decision tree.

A careful analysis of these arguments shows that they rely on some further assumptions and therefore may not be prove that individual decision makers must follow both RCLA and CIA (see Machina [20] and McClennen [21] for arguments regarding Markowitz and Raiffa's support of dynamic consistent decision rules and Border and Segal [2, 3] for an analysis of Dutch books involving violations of probability theory). But even if the Dutch books are valid, they can hardly be understood as practical arguments. At best they are theoretical arguments that can be used to persuade a reluctant decision maker to follow expected utility theory. But it is certainly conceivable to imagine willingness to pay a hypothetical price to satisfy an intuitive feelings regarding the proper simplification of a two stage lottery.

The example presented in the introduction is an extended version of the Dutch book argument and seems to weigh against violations of RCLA and CIA performed by the public official. It is hard to justify mathematical mistakes done by such officials, and they will have hard time explaining why they chose an option where all possible outcomes are inferior to an alternative option. Theorems 1 and 2 show that none of these arguments can be raised in a large society. For sufficiently large n, the official's decisions are consistent with both RCLA and CIA.

0, and such that for all $Y \in \mathcal{X}$, $\mathbf{E}[v(Y)] \geqslant (Y) \geqslant \mathbf{E}[v(Y)] - v(\bar{\ }) +$,
where $v(\ ) := (\ _z)$ and $^{-}$ is the highest possible outcome in Y.

Note that if the preferences $\succeq$ with are as in the definition of wrapping, then so are $\succeq$ with + + (1 − +) for all . We therefore assume wlg that $v(0) = (\ _0) = 0$.

Recall that $a_v = \lim_{x \to -\infty} - \ ^{v}$

as (Y), the certainty equivalent of Y. Let $b := v(\bar{x})$. By the wrapping assumption there exist $\geqslant 1$, $\geqslant 0$, and such that

$$
\begin{aligned}
(Y) &\geqslant \mathbb{E}[v(Y)] - v(\bar{\ }) + \\
&\geqslant \mathbb{E}[v(Y)] - v(n\bar{x}) + \qquad (9)\\
&\geqslant \mathbb{E}[v(Y)] - nb +
\end{aligned}
$$

where the last inequality follows from the concavity of v and the fact that $v(0) = 0$. Hence, using inequality (9) and the fact that the highest possible outcome in $^{n}_{\mathrm{CI}}$ cannot exceed $n\bar{x}$, we get

$$
\begin{aligned}
(\ ^{n}_{\mathrm{CI}}) &\geqslant \mathbb{E}[v(\ ^{n}_{\mathrm{CI}})] - nb + \\
&= \sum_j \ _{nj} v(\ (Y_{nj})) - nb + \\
&= \sum_j \ _{nj} \ (Y_{nj}) - nb + \\
&\geqslant \sum_j \ _{nj}[\ \mathbb{E}[v(Y_{nj})] - nb + \ ] - nb + \\
&= \ ^2\mathbb{E}[v((\ ^{n})_{\mathrm{R}})] - (\ +1) nb + (\ +1) \\
&= \ ^2\mathbb{E}[v(\ ^{n}_{\mathrm{R}})] - (\ +1) nb + (\ +1)
\end{aligned}
$$

Here too, the last two equality signs hold since the expected utility model satisfies RCLA and by Claim 5. Since $(\ ^{n}_{\mathrm{CI}}) = v(\ (\ ^{n}_{\mathrm{CI}}))$, we get

$$
v(\ (\ ^{n}_{\mathrm{CI}})) \geqslant \ ^2 v(\ _v(\ ^{n}_{\mathrm{R}})) - (\ +1) nb + (\ +1) \qquad (10)
$$

Next we show that $\lim_{n\to\infty} \frac{c(Q^{n}_{\mathrm{CI}})}{n}$ exists and is equal to

$\geqslant$ 0 such that $(Y) \geqslant \mathbb{E}[v(Y)] - v($

Proof of Claim 2: Since w 0, the requirement $\frac{w'}{w} \leqslant 0$ is equivalet

Proof of Claim 3: To see that $\mathbf{E}[v(Y)] \geqslant \ (Y)$, note that similarly to the proof of claim 2, in the DA model the utilities of all the outcomes that are strictly preferred to Y are multiplied by $\frac{1}{1+bF_{\mathbf{Y}}(c(Y))} \leqslant 1$, while all other utilities are multiplied by $\frac{1+b}{1+bF_{\mathbf{Y}}(c(Y))} \geqslant 1$ (note that $\ (t \ b \ (Y))\mathrm{d}F_Y(t) = 1$). To show that there exist $\ \geqslant 1$ and $\ \geqslant 0$ such that $(Y) \geqslant \ \mathbf{E}[v(Y)] - \ v(\bar{\ })$, observe that $(t \ b \ (Y)) \leqslant 1 + b$ and proceed as in the RDU model with $\ = 1 + b$ and $\ = b$. ■

Proof of Claim 4: We consider wlg finite lotteries of the form $Y = (\ _1 \ \frac{1}{n}; \ ; \ _n \ \frac{1}{n})$ where $\ _1 \leqslant \ \leqslant \ _n$ and start with the inequality $(Y) \leqslant \mathbf{E}[v(Y)]$. Since $v(\ ) = \ (\ )$, we get

$$
\begin{aligned}
(Y) &= \frac{1}{n^2} \sum_i (\ _i \ _i) + \sum_i \sum_{j>i} [\ (\ _i \ _j)) + (\ _j \ _i)] \\
&\leqslant \frac{1}{n^2} \sum_i v(\ _i) + \sum_i \sum_{j>i} [v(\ _i) + v(\ _j)] \\
&= \frac{1}{n^2} \sum_i v(\ _i) + (n-1) \sum_i v(\ _i) \quad = \mathbf{E}[v(Y)]
\end{aligned}
$$

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where the inequality follows by the condition $(x \ x) + (\ ) \geqslant (x \ (Y)) \leqslant \mathbf{E}[v(Y)]$

$$\geqslant 2\mathbf{E}[v(Y)] - \frac{1}{n^2} \sum_{j=1}^{n} (2 - 1) v( )$$

$$= 2\mathbf{E}[v(Y)] - v( )$$

The first inequality follows by the fact that for , $_i \geqslant _j$ and by the monotonicity of

It thus follows that w is more risk averse than all $u \in \mathcal{W}$, hence for all such u, ${}_w(Y_j^n) \leqslant {}_u(Y_j^n)$ and similarly to the argument above,

$$
\begin{aligned}
{}_w(\ {}^n_{\mathbf{CI}}) &= {}_w(\ {}_w(Y_{n1})\ \ {}_{n1};\ \ ;\ {}_w(Y_{nH})\ \ {}_{nH}) \\
&\leqslant {}_w(\ {}_u(Y_{n1})\ \ {}_{n1};\ \ ;\ {}_u(Y_{nH})\ \ {}_{nH}) \\
&\leqslant {}_u(\ {}_u(Y_{n1})\ \ {}_{n1};\ \ ;\ {}_u(Y_{nH})\ \ {}_{nH}) = {}_u(\ {}^n_{\mathbf{CI}})
\end{aligned}
$$

It thus follows that ${}_w(\ {}^n_{\mathbf{CI}}) \leqslant \min_{u \square \mathbf{W}} {}_u(\ {}^n_{\mathbf{CI}}) = (\ {}^n_{\mathbf{CI}}) = (\ {}^n_{\mathbf{CI}})$ (recall that $(\ {}_z) = \ )$. Let satisfy ${}_{\mathcal{W}}(\ ) = \mathrm{E}[\ {}_{\mathcal{W}}(\ {}_{\mathbf{R}})]$. By [26, Lemma 6], $\lim_{n \to \infty} \frac{c_{\mathbf{W}}(Q^{\mathbf{n}}_{\mathbf{R}})}{n} =$ and hence, since ${}_w(\ {}^n_{\mathbf{R}}) = {}_w(\ {}^n_{\mathbf{CI}}) \leqslant (\ {}_{(\mathrm{CY}}{}^n$

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