REAL ANALYSIS QUALIFYING EXAM

MAY 2017

Answer all 4 questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Exercise 1. (25 points). Suppose $(\mathbb{R}, \mathcal{A}, \mu)$ is the measure space with $\mathcal{A}$ the σ-algebra of all subsets and μ is the counting measure (i.e., $\mu(A) = \#A$ for finite sets and $\mu(A) = \infty$ for infinite sets)

(1) Is the function

$$F(x) = \begin{cases} e^{-|x|}, & x \quad \mathbb{Q} \\ 0, & x \quad \mathbb{Q} \end{cases}$$

integrable? (Prove or disprove.)

(2) Is the set of finitely supported functions dense in $L^1(\mathbb{R}, \mathcal{A}, \mu)$? (Prove or disprove.)

Exercise 2. (25 points).

The total variation of a complex measure ν is the positive measure $|\nu|$ defined by the property that if $d\nu = f d\mu$ for a positive measure μ and $f \quad L^1(\mu)$, then $d|\nu| = |f| d\mu$. Show that this is well defined by showing that

(1) There always exists such a positive measure μ.
(2) The definition does not depend on the choice of μ.

Exercise 3.

COMPLEX ANALYSIS QUALIFYING EXAM

Write your answers on the test pages. Show all your work and explain all your reasoning. You may use any result from class or the course notes, as long as you state clearly what result you are using (including its hypotheses). Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

Name: ____

Date: June 8, 2017.

1. (10 points) Let $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Are D and A conformally equivalent? Justify your answer.

2. (10 points) Evaluate the following integral:

$$\int_0^\infty \frac{dx}{x^4 + 1}.$$

3. (10 points) Suppose f and g are entire functions such that $|f(z)| < |g(z)|$ for all z with $|z| \geq R$ for a constant R. Prove that f/g is a rational function.

4. (10 points) Let X be a closed and connected Riemann surface.

(a) Let ω be a meromorphic one-form on X. Prove that the sum of residues of ω is zero.

(b) Given $n \geq 2$ distinct points $p_1, \ldots, p_n \in X$ and a tuple $r_1, \ldots, r_n \in \mathbb{C}$ such that $\sum_{i=1}^{n} r_i = 0$, prove that there exists a meromorphic one-form ω such that ω is holomorphic on $X \setminus \{p_1, \ldots, p_n\}$ and has a simple pole at p_i with residue r_i for each i.