Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

Question 1. Let $f : [0;1]$! R be a nonnegative Lebesgue measurable function such that $f > 0$ almost everywhere. Prove that for any > 0, there exists > 0 such that for any Lebesgue measurable subset S $[0;1]$ with $m(S) >$, we have $\int_S f \, dm >$

Question 4. Let $(X; \| \ \|)$ be a normed ($\mathbb{R}$-)linear space and let $(X ; \| \ \|_{op})$ denote its dual Banach space of (real-valued) bounded linear functions (equipped with the operator norm). Prove that the linear map $: X \to X$ given by

$$(x)(f) = f(x)$$

is an isometry.

(You may use without proof the fact that for each $x \in X$ there exists $f \in X$ such that $\|f\|_{op} = 1$ and $\|x\| = f(x)$.)