

Algebra Qualifying Exam

Fall 2014

You have 3 hours to answer all questions.

1. Determine the number of conjugacy classes of elements of order 4 in $GL_4(\mathbb{C})$, and give a representative of each class. Do the same for $GL_4(\mathbb{F}_7)$.

2. Find the Galois groups of $f(x) = x^5 + 7x^3 + 6x^2 + x + 5$ over \mathbb{F}_2 , \mathbb{F}_3 , \mathbb{F}_5 , and \mathbb{Q} . You may assume without proof that $f(x) \in \mathbb{F}_3[x]$ has no irreducible quadratic factors.

3. Let R be a Noetherian ring. For any ideal $J \subseteq R$ define

$$P_J = \{x \in R : x^k \in J \text{ for some } k \in \mathbb{Z}^+\}$$

If $P_J = J$, show that J can be expressed as a finite intersection of prime ideals. Hint: among all counterexamples, a maximal one cannot be prime.

4. Classify the groups of order $2915 = 5 \cdot 11 \cdot 53$.

5. Suppose R is a PID and A and B are R -modules. Let $B_{\text{tors}} \subseteq B$ be the submodule of R -torsion elements. Prove that

$$\text{Tor}_1^R(A; B) = \text{Tor}_1^R(A; B_{\text{tors}}):$$

6. Suppose R is a Noetherian ring and $\mathfrak{p} \subseteq R$ is a prime ideal. Show that there is an $r \notin \mathfrak{p}$ such that $S^{-1}R \rightarrow R_{\mathfrak{p}}$ is injective, where $S = \{1, r, r^2, r^3, \dots\}$.

7. Suppose R is a commutative local ring, and M and N are R -modules satisfying

$$M \otimes_R N = 0:$$

(a) If M and N are finitely generated, show that either $M = 0$ or $N = 0$.

(b) Show by example that (a) is false if we drop the hypothesis that M and N are finitely generated.

8. Let $\zeta_8 \in \mathbb{C}$ be a primitive eighth root of unity. The ring of integers in $\mathbb{Q}[\zeta_8]$ is $\mathbb{Z}[\zeta_8]$ (you may assume this without proof). If p is a prime, determine the number of primes of $\mathbb{Z}[\zeta_8]$ above p when

(a) $p = 2$,

(b) $p \equiv 1 \pmod{8}$,

(c) $p \equiv 3, 5, 7 \pmod{8}$.