Real Analysis Qual

June 10, 2016

Problem 1. Let $\sim$ be the equivalence relation on the interval $[0, 1]$ given by $x \sim y$ iff $x - y \in \mathbb{Q}$. Choose one element from each equivalence class (using the Axiom of Choice). Let $A \subset [0, 1]$ denote the set of these chosen elements. For a given set $B \subset \mathbb{R}$, define $B + x = \{y + x \mid y \in B\}$.

(a) Show that the sets $A + q$, for $q \in \mathbb{Q} \cap [-1, 1]$, are disjoint.

(b) Show that $[0, 1] \subset \bigcup_{q \in \mathbb{Q}\cap[-1,1]}(A + q) \subset [-1, 2]$.

(c) Show that A

(c) Prove Fatou's Lemma, which states that

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

for any sequence (f_n) of non-negative, measurable functions. To get started, let $g_n = \inf_{i \geq n} f_i$. Observe that, with

$\mu(X) < \infty$. Fix $E_1, \ldots, E_n \in A$ and $c_1, \ldots, c_n \in \mathbb{R}_{\geq 0}$. Define $\nu : A \to [0, \infty]$ by

$$\nu(A) = \sum_{i=1}^{n} c_i \mu(A \cap E_i).$$

(a) Show that ν is a measure.

(b) Show that ν is absolutely continuous with respect to μ.

Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.
Notation: $\mathbb{D}$ = open unit disk, $\mathbb{C}^* = \mathbb{C} - \{0\}$, $\mathbb{H}$ = upper half plane.

1) Find a holomorphic function f on $\mathbb{C}^*$ such that

   - f is a pointwise limit of polynomial functions, but
   - f is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to f uniformly on compact subsets of $\mathbb{C}^*$).

   Prove both assertions for your choice of f.

2) Find a biholomorphism between $\mathbb{H}$ and the region

$$U = \{z \in \mathbb{C} \,|\, |z-1| < 1, |z-i| < 1\}.$$

   It is enough to write down explicitly functions whose composition yields a biholomorphism from $\mathbb{H}$ to U or from U to $\mathbb{H}$.

3) Let $U \subset \mathbb{C}$ be a simply connected region. For any point $a \in U$, the Green function of U

where $\wp(z)$ is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Justify carefully any techniques you use.